SI 424 - Tutorial

Tutorial -1 :

Ex: Suppose (X,Y) random vector joint pdf is

$$f(x,y)=\begin{cases} c & ; |x|+|y|<1 \\ 0 & ; |x|+|y|\geq 1 \end{cases}$$

a) find c

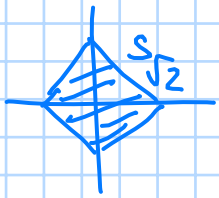
b) fix $x_0 \in (-1,1)$ compute conditional pdf of Y given $X=x_0$

Ans:

a) as f is density

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \iint_S c\,dx\,dy = 1$$

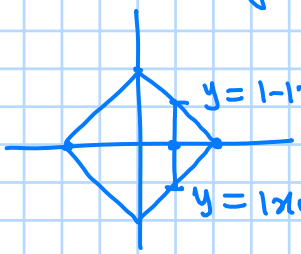


area of $S = (\sqrt{2})^2 = 2$

$$= \text{area of } \{(x,y)\in\mathbb{R}^2 \mid |x|+|y|<1\}$$

$$\Rightarrow 2c = 1 \Rightarrow c = \frac{1}{2}$$

b) fix $x_0 \in (-1,1) = \text{Range}(X)$



conditional range of Y given $X = x_0$ is $(|x_0|-1, 1-|x_0|)$

$y \notin (|x_0|-1, 1-|x_0|)$ then the

conditional pdf $f_{Y|X=x_0}(y)$ of Y given $X=x_0$ takes value 0

$y \in (|x_0|-1, 1-|x_0|)$

$$f_{Y|X=x_0}(y) = \frac{f(x_0,y)}{f_X(x_0)}$$

f_X is marginal pdf of X

Note that for each $x_0 \in (-1,1) = \text{range}(X)$

$$f_X(x_0) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{|x_0|-1}^{1-|x_0|} \frac{1}{2}\,dy = 1-|x_0|$$

So, for all $y \in (|x_0|-1, 1-|x_0|)$ $f_{Y|X=x_0}(y) = \frac{f(x_0,y)}{f_X(x_0)} = \frac{1/2}{1-|x_0|}$

$$= \frac{1}{2(1-|x_0|)}$$

Therefore, the required conditional pdf is (uniform distribution)

$$f_{Y|X=x_0}(y) = \begin{cases} \frac{1}{2(1-|x_0|)} & ; y \in (|x_0|-1, 1-|x_0|) \\ 0 & ; \text{otherwise} \end{cases}$$

So, $Y|X=x_0 \sim \text{Unif}(|x_0|-1, 1-|x_0|)$

Exe: $X_1, \dots X_n \overset{iid}{\sim} \text{Poi}(\lambda)$

$s_0 \in \mathbb{N} \cup \{0\}$

$S = \sum_{i=1}^{n} X_i$

a) find conditional pmf of X_1 given $S=s_0$

b) compute $\mathbb{E}[X_1|S]$

Ans: a) $S = \sum_{i=1}^{n} X_i$ $\quad S \sim \text{Poi}(n\lambda)$

Range $S = \mathbb{N} \cup \{0\}$

$s_0 \in \mathbb{N} \cup \{0\}$ to find conditional pmf

X_1 given $S=s_0$

$s_0 = 0$ then $X_1 \cong 0$

let's assume $s_0 \in \mathbb{N}$

the conditional range of X_1 given $S=s_0 = \{0, \dots s_0\}$

$f_{X_1|S=s_0}(x)$ will be 0 if $x \notin \{0, \dots s_0\}$

now for $x \in \{0, \dots s_0\}$

$$f_{X_1|S=s_0}(x) = \frac{P[X_1=x, S=s_0]}{P[S=s_0]}$$

$$= \frac{P[X_1=x, \sum_{i=2}^{n} X_i = s_0 - x]}{P[S=s_0]}$$

$$= \frac{P[X_1=x]\, P[\sum_{i=2}^{n} X_i = s_0 - x]}{P[S=s_0]}$$

as $X_1, X_2, \dots X_n$ are iids

$$\Rightarrow X_1 \perp\!\!\!\perp \sum_{i=2}^{n} X_i \sim \text{Poi}((n-1)\lambda), \quad X_1 \sim \text{Poi}(\lambda)$$

$$S \sim \text{Poi}(n\lambda)$$

$$= \frac{\frac{e^{-\lambda}\lambda^x}{x!_0}\, e^{-(n-1)\lambda} \frac{((n-1)\lambda)^{s_0-x}}{(s_0-x)!_0}}{\frac{e^{-n\lambda}(n\lambda)^{s_0}}{s_0!_0}}$$

$$= \frac{(n-1)^{s_0-x}}{(n)^{s_0}} \frac{s_0!_0}{x!_0 (s_0-x)!_0} = \frac{(n-1)^{s_0-x}}{(n)^{s_0}} \binom{s_0}{x}$$

$$= \binom{s_0}{x}\left(\frac{1}{n}\right)^x\left(1-\frac{1}{n}\right)^{s_0-x}$$

the required conditional pmf is

$$f_{X_1|S=s_0}(x) = \begin{cases} \binom{s_0}{x}\left(\frac{1}{n}\right)^x\left(1-\frac{1}{n}\right)^{s_0-x} & ; x \in \{0,1,\ldots s_0\} \\ 0 & ; x \notin \{0,1,\ldots s_0\} \end{cases}$$

also valid when $s_0 = 0$

$X_1|S=s_0 \sim Bin\left(s_0, \frac{1}{n}\right)$

with $Bin\left(0, \frac{1}{n}\right) \cong 0$

b) $\mathbb{E}[X|S=s_0] = s_0 \cdot \frac{1}{n} \quad \forall s_0 \in \mathbb{N} \cup \{0\} = Range(S)$

$(\because$ Binomial$)$

$\mathbb{E}[X|S]$ takes value $\frac{s_0}{n}$ when $S = s_0$

$\therefore, \mathbb{E}[X|S] = \frac{S}{n} \Rightarrow \mathbb{E}[X_1 | \sum_{i=1}^{n} X_i] = \frac{\sum_{i=1}^{n} X_i}{n}$

Tutorial-2:

Ex: Suppose $X_1, X_2, \ldots X_n \overset{iid}{\sim}$ Exponential distribution with mean 1, find pdf of j^{th} order statistic $X_{(j)}$, $j \in [n]$

Ans:

Now, pdf of $X_{(j)}$ $\quad f_{X_{(j)}} = \binom{n}{j} (j) (F(x))^{j-1} (1-F(x))^{n-j} f(x)$

$\lambda = 1$ Exp(1) i.e

$$f_X(x) = \lambda e^{-\lambda x} = e^{-x}$$

$$\text{then } f_{X_{(j)}} = \binom{n}{j} (j) \left(\int_0^x e^{-x} dx \right)^{j-1} \left(1 - \int_0^x e^{-x} dx \right)^{n-j} e^{-x}$$

$$= \binom{n}{j} (j) \left(\left| \frac{e^{-x}}{-1} \right|_0^x \right)^{j-1} \left(1 - \left| \frac{e^{-x}}{-1} \right|_0^x \right)^{n-j} e^{-x}$$

$$= \binom{n}{j} (j) \left(\frac{e^{-x}}{-1} + 1 \right)^{j-1} \left(1 - \left(\frac{e^{-x}}{-1} + 1 \right) \right)^{n-j} e^{-x}$$

$$= \binom{n}{j} (j) \left(1 - e^{-x} \right)^{j-1} \left(e^{-x} \right)^{n-j} e^{-x}$$

$$= \binom{n}{j} (j) (1-e^{-x})^{j-1} (e^{-x})^{n-j+1}$$